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STABILITY OF JETS OF AN IDEAL PONDERABLE LIQUID

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The stability of jets of an ideal liquid was investigated in [1-4], where it was assumed that the undisturbed flow is parallel, and the velocity of the liquid in the jet is constant. In this paper we examine the stability of jets of ponderable liquids within the framework of linear theory, taking into account the effect of the surrounding medium, which is also assumed to be ideal. The ponderability of the liquid is manifested in the deviation of the jet boundaries from the parallel direction and the dependence of the velocity on the longitudinal coordinate. These features can be taken into account as, for instance, in the theory of stability of laminar boundary layers, where the flow is assumed to be quasi-parallel. In this case the dependence of the jet thickness and velocity in the jet on the longitudinal coordinate can be regarded as parametric. In this paper we examine a significantly nonparallel flow and, hence, for determination of the stability characteristics of a jet flow in this case we propose an asymptotic method.

1. Basic Equations. The basic equations have the form

$$\begin{gathered}\frac{\partial^2\Phi_i}{\partial x^2}+\frac{k}{r}\frac{\partial\Phi_i}{\partial r}+\frac{\partial^2\Phi_i}{\partial r^2}=0,\qquad i=1,\,2,\\ \frac{\partial\Phi_i}{\partial t}+\frac{p_i}{\rho_i}+\frac{u_i^2+v_i^2}{2}\mp gx=\mathrm{const}_i,\end{gathered}\tag{1.1}$$

where $u_i=\partial\Phi_i/\partial x$, $v_i=\partial\Phi_i/\partial r$ are the projections of the velocity on the x and r axes, p_i is the pressure, ρ_i is the density; k = 0 for a plane jet, k = 1 for an axisymmetric jet; the subscript 1 relates to the flow parameters in the jet, and the subscript 2 relates to the surrounding medium. On the jet boundary the conditions

$$v_i=\partial a/\partial t+u_i\partial a/\partial x,\ p_1-p_2=\sigma(1/R+k/a),$$
$$R=-\frac{[1+(\partial a/\partial x)^2]^{3/2}}{\partial^2 a/\partial x^2}$$

are fulfilled, where a is the radius (k = 1) or halfwidth (k = 0) of the jet; σ is the coefficient of surface tension.

Henceforth, we will deal with the problem in region 1 in the variables $\xi=x/a_0$, $\tau=Ut/a_0$, $n=r/a_0$, and in region 2 in the variables ξ, τ, and $N=(r-a)/a_0\xi^m+k$, where a_0 is the linear scale; U is the velocity scale; m is a coefficient which will be determined below. Keeping within the framework of linear theory, we put the solutions of Eqs. (1.1) in the form

$$\Phi_i=a_0U(\varphi_i+\varphi_{i\delta}),\quad p_i=\rho_1U^2(P_i+p_{i\delta}),\quad a/a_0=y_*+\delta,$$

where the first terms on the right-hand sides correspond to undisturbed motion, and the second terms to disturbed motion.

In the new variables the equations for the disturbed motion and the boundary equations have the form (the velocity of the surrounding medium is zero)

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$$\frac{\partial^2\varphi_{1\delta}}{\partial\xi^2}-\left(\frac{y_*''}{y_*}-2\frac{y_*'^2}{y_*^2}\right)n\frac{\partial\varphi_{1\delta}}{\partial n}-\left(\frac{\delta''}{y_*}-\frac{y_*''\delta}{y_*^2}-4\frac{y_*'\delta'}{y_*^2}+4\frac{y_*'^2\delta}{y_*^3}\right)n\frac{\partial\varphi_1}{\partial n}-$$

$$-2\frac{y_*'}{y_*}n\frac{\partial^2\varphi_{1\delta}}{\partial\xi\partial n}-2\left(\frac{\delta'}{y_*}-\frac{y_*'\delta}{y_*^2}\right)n\frac{\partial^2\varphi_1}{\partial\xi\partial n}+\left(2\frac{y_*'\delta'}{y_*^2}n^2-4\frac{y_*'^2\delta}{y_*^3}n^2-2\frac{\delta}{y_*^3}\right)\frac{\partial^2\varphi_1}{\partial n^2}+$$

$$+\frac{k}{n}y_*^{-2}\frac{\partial\varphi_{1\delta}}{\partial n}-2\frac{k}{n}\frac{\delta}{y_*^3}\frac{\partial\varphi_1}{\partial n}+\left(\frac{y_*'^2}{y_*^2}n^2+y_*^{-2}\right)\frac{\partial^2\varphi_{1\delta}}{\partial n^2}=0,$$

$$\frac{\partial\varphi_{1\delta}}{\partial\tau}-\frac{\delta^{\cdot}}{y_*}\frac{\partial\varphi_1}{\partial n}n+\frac{p_1}{\rho_1}+\left(\frac{\partial\varphi_1}{\partial\xi}-\frac{y_*'}{y_*}n\frac{\partial\varphi_1}{\partial n}\right)\left[\frac{\partial\varphi_{1\delta}}{\partial\xi}-\frac{y_*'}{y_*}\frac{\partial\varphi_{1\delta}}{\partial n}-\right.$$

$$\left.-\left(\frac{\delta'}{y_*}-\frac{y_*'\delta}{y_*^2}\right)n\frac{\partial\varphi_1}{\partial n}\right]+y_*^{-2}\frac{\partial\varphi_1}{\partial n}\left(\frac{\partial\varphi_{1\delta}}{\partial n}-\frac{\delta}{y_*}n\frac{\partial\varphi_1}{\partial n}\right),\quad y_*'=\frac{dy_*}{d\xi},\quad \delta^{\cdot}=\frac{\partial\delta}{\partial\tau};\qquad (1.2)$$

$$\frac{\partial^2\varphi_{2\delta}}{\partial\xi^2}+2m\xi^{-2}N\frac{\partial\varphi_{2\delta}}{\partial N}+\frac{y_*'}{\xi^{m+1}}2m\frac{\partial\varphi_{2\delta}}{\partial N}-\frac{y_*''}{\xi^m}\frac{\partial\varphi_{2\delta}}{\partial N}-2m\xi^{-1}N\frac{\partial^2\varphi_{2\delta}}{\partial\xi\partial N}-$$

$$-2\frac{y_*'}{\xi^m}\frac{\partial^2\varphi_{2\delta}}{\partial\xi\partial N}+m^2\xi^{-2}N^2\frac{\partial^2\varphi_{2\delta}}{\partial N^2}+\left(\frac{y_*'}{\xi^{m+1}}N+\xi^{-2m}\right)\frac{\partial^2\varphi_{2\delta}}{\partial N^2}=0,$$

$$p_{2\delta}+\frac{\rho_2}{\rho_1}\frac{\partial\varphi_{2\delta}}{\partial\tau}=0;\qquad (1.3)$$

$$y_*^{-1}\left(\frac{\partial\varphi_{1\delta}}{\partial n}-\frac{\delta}{y_*}\frac{\partial\varphi_1}{\partial n}\right)\Big|_{n=1}=\delta^{\cdot}+\left(\frac{\partial\varphi_1}{\partial\xi}-\frac{y_*'}{y_*}\frac{\partial\varphi_1}{\partial n}\right)\Big|_{n=1}\delta'+\left[\frac{\partial\varphi_{1\delta}}{\partial\xi}-\right.$$

$$\left.-\frac{y_*'}{y_*}\frac{\partial\varphi_{1\delta}}{\partial n}-\left(\frac{\delta'}{y_*}-\frac{y_*'\delta}{y_*^2}\right)\frac{\partial\varphi_1}{\partial n}\right]\Big|_{n=1}y_*',\quad \xi^{-m}\frac{\partial\varphi_{2\delta}}{\partial N}\Big|_{N=k}=\delta^{\cdot},$$

$$p_{1\delta}\big|_{n=1}-p_{2\delta}\big|_{N=k}=-\left(\delta''+\frac{k}{y_*^2}\delta\right)\mathrm{We}^{-1},\quad \frac{\partial\varphi_{1\delta}}{\partial n}\Big|_{n=0}=0.\qquad (1.4)$$

2. Plane Jet (k = 0). In this case the asymptotic form of the solution (large ξ) of the undisturbed equations is fairly simple:

$$\varphi_1=C\left(\xi^{3/2}-\frac{3}{8}n^2\xi^{-3/2}\right),$$

$$y_*=\xi^{-1/2},$$

$$C=\begin{cases}\left(\dfrac{a_0g}{U^2}\right)^{1/2}, & \dfrac{\rho_2}{\rho_1}\ll 1,\\[2ex] \left(\dfrac{a_0g}{U^2}\dfrac{\rho_2}{\rho_1}\right)^{1/2}, & \dfrac{\rho_2}{\rho_1}\gg 1.\end{cases}$$

We will not consider the subsequent terms of the expansions, since their order lies outside the number of approximations considered in this paper. To determine the solutions of the above equations we put the first terms of the expansions of functions $\varphi_{i\delta}$, $p_{i\delta}$, and δ in the form

$$\delta\sim\xi^p\chi,\ \varphi_{1\delta}\sim\xi^s\Omega_1(n)\chi,\ \varphi_{2\delta}\sim\xi^\alpha\Omega_2(N)\chi,$$
$$p_{1\delta}\sim\xi^\beta\chi,\ p_{2\delta}\sim\xi^\zeta\chi,\ \chi\sim\exp\left[j(\omega\tau+\gamma\xi^r)\right],\qquad (2.1)$$

where ω is the frequency, γ is the wave number; p, r, s, α, and β are coefficients. Substituting expressions (2.1) in the kinematic conditions (1.4), equating the orders of the first three terms in the first condition, and retaining the terms in the second, we obtain [for the pressure we use the second equations of (1.2) and (1.3)]

$$r=1/2,\ s=\beta=p+3/2,\ \alpha=\zeta=p+m.$$

The value of m can be found from the condition that, in view of the boundedness of the potential in the external region at infinity, we must retain terms with second derivatives with respect to ξ and N in the first approximation of (1.3). Since $r = {}^1/_2$ we have $m = {}^1/_2$, as a result of which the equation for Ω_2 after the terms of higher order have been discarded has the form

$$d^2\Omega_2/dN^2 - \gamma^2\Omega_2/4 = 0, \tag{2.2}$$

and $\Omega_2 = D\exp\left(-\frac{\gamma}{2}N\right)$, i.e., when $N \to \infty$, $\Omega_2 \to 0$. We now write out the complete asymptotic expansions of the solutions

$$\delta = A\xi^p\chi,\ \varphi_{1\delta} = \xi^{p+3/2}(\Omega_{10} + \xi^{-1/2}\Omega_{11} + \ldots)\chi,$$
$$\varphi_{2\delta} = \xi^{p+1/2}(\Omega_{20} + \xi^{-1/2}\Omega_{21} + \ldots)\chi,\ p_{1\delta} = \xi^{p+3/2}(R_{10} + \xi^{-1/2}R_{11} + \ldots)\chi,$$
$$p_{2\delta} = \xi^{p+1/2}(R_{20} + \xi^{-1/2}R_{21} + \ldots)\chi,\ \chi = \exp\,[j(\omega\tau + \gamma_0\xi^{1/2} + \gamma_2\xi^{-1/2} + \ldots)].$$

After substituting the above expressions in the basic equations (1.2), (1.3) we obtain a system of simple equations [in the inner region of the form $d^2\Omega_{1j}/dn^2 = F(\Omega_{1j-4})$, in the outer region of form (2.2) with known right-hand sides]. Omitting the intermediate operations, we write equations for γ_j and p

$$\gamma_0 = -\frac{4}{3}C^{-1}\omega,\quad p_{\mathrm{I,II}} = -\frac{3}{4} \pm \sqrt{\frac{1}{16} + \frac{8}{27}C^{-3}\frac{\rho_2}{\rho_1}\omega^3},$$
$$\gamma_2 = -\left\{\frac{1}{2}C\left(p+\frac{1}{2}\right)^2\omega^{-1} + \frac{8}{27}C^{-2}\frac{p+\frac{1}{4}}{p+\frac{1}{2}}\omega^2\right\}. \tag{2.3}$$

If we now relate the magnitude of the disturbances to the halfwidth of the undisturbed jet, we will have

$$\varepsilon = \delta/y_* = A\exp\{j\,[\omega\tau + \gamma_0\xi^{1/2} - j(p+1/2)\ln\xi + \gamma_2\xi^{-1/2} + \ldots]\}.$$

Since p has two roots, one of which is such that

$$p_{\mathrm{I}} + \frac{1}{2} = -\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{8}{27}C^{-3}\frac{\rho_2}{\rho_1}\omega^3} > 0$$

at any frequency $\omega > 0$, we can conclude that the relative disturbances arising in the initial portion of a jet of heavy liquid increase with increasing distance from the source. The increase in relative disturbances is given by a power relation $\varepsilon \sim \xi^{p_{\mathrm{I}}+1/2}$. The main factor in the instability of a plane jet is the surrounding medium. Equation (2.3) for p does not contain We^{-1}, since the terms which take this quantity into account have a high order of smallness. When $\rho_2/\rho_1 = 0$ the plane jet is stable.

3. Axisymmetric Jet (k = 1). For an axisymmetric jet the asymptotic solution can be bounded by the expressions

$$\varphi_1 = C\xi^{3/2},\quad y_* = \xi^{-1/4}.$$

In this case an evaluation of the orders of the first terms of the solution expansions gives $r = 1/2$, $s = \beta = p + 5/4$, $\alpha = \zeta = p + 1/2$. To obtain nontrivial regular expansions we need the following expressions:

$$\delta = A\xi^p\chi,\ \varphi_{1\delta} = \xi^{p+5/4}(\Omega_{10} + \xi^{-1/8}\Omega_{11} + \xi^{-2/8}\Omega_{12} + \ldots)\chi,$$
$$\varphi_{2\delta} = \xi^{p+1/2}(\Omega_{20} + \xi^{-1/8}\Omega_{21} + \ldots)\chi,\ p_{1\delta} = \xi^{p+5/4}(R_{10} + \xi^{-1/8}R_{11} + \ldots)\chi,$$
$$p_{2\delta} = \xi^{p+1/2}(R_{20} + \xi^{-1/8}R_{21} + \ldots)\chi,$$
$$\chi = \exp\,[j(\omega\tau + \gamma_0\xi^{4/8} + \gamma_1\xi^{3/8} + \gamma_2\xi^{2/8} + \gamma_3\xi^{1/8} + \ldots)]. \tag{3.1}$$

The system of equations obtained after substitution of (3.1) in (1.2) and (1.3), as in the plane case, is fairly simple and allows direct integration, as a result of which we can obtain

$$\gamma_0 = -\frac{4}{3}\omega C^{-1},\quad \gamma_1 = \gamma_2 = 0,$$

$$\gamma_{3\mathrm{I,II}} = \pm j \frac{16}{9} \frac{\omega}{C^2} \left[2\,\mathrm{We}^{-1} + 3\omega C \frac{K_0\left(\frac{|\gamma_0|}{2}\right)}{\left|K_0'\left(\frac{|\gamma_0|}{2}\right)\right|} \frac{\rho_2}{\rho_1} \right]^{1/2}, \qquad p = -\frac{9}{16}, \quad \gamma_{5\mathrm{I,II}} = \frac{15}{8} \gamma_{3\mathrm{I,II}}^{-1}, \tag{3.2}$$

where K_0 is a Bessel function of the second kind of imaginary argument. The behavior of the disturbances can be assessed from γ_3. Since Im $\gamma_{3\mathrm{II}} < 0$ and $\omega > 0$, then in this case too the disturbances arising in the initial portion increase with increase in ξ according to the relation

$$\varepsilon = \delta/y_* \sim \exp j\left(\gamma_{3\mathrm{II}}\xi^{1/8}\right).$$

As distinct from a plane jet, We^{-1} here affects the development of disturbances at large ξ. Thus, as (2.3) and (3.2) show, jets of heavy liquid are unstable at any frequencies $\omega > 0$; with increase in ω the disturbances along the jet grow more rapidly, which leads to reduction of the length of the intact part of the jet. This is qualitatively consistent with the results of experimental investigations of the development of unstable disturbances over the surface of capillary liquid jets flowing vertically downward ($\rho_1/\rho_2 \gg 1$). For instance, in [5, 6] it was reported that when velocity fluctuations are imposed the effective length of the jet region (the portion in which unstable disturbances develop from a small, but experimentally determinable, amplitude before breakup) decreases with increase in frequency of the imposed disturbance.

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